FOLIATED GROUP ACTIONS AND CYCLIC COHOMOLOGY

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ABSTRACT. In this paper, we prove a cyclic Lefschetz formula for foliations. To this end, we define a notion of equivariant cyclic cohomology and show that its expected pairing with equivariant K-theory is well defined. This enables to associate to any Haefliger's transverse invariant current on a compact foliated manifold, a Lefschetz formula for leafwise preserving isometries.

CONTENTS

1. INTRODUCTION

Let (V, g) be a smooth riemannian compact manifold. Let f be any isometry of (V, g) and let H be the compact Lie group generated by f in the group of isometries $Iso(V, g)$. Then, to any H-invariant elliptic complex (E, d) over V, one associates a Lefschetz number $L(f; E, d)$ defined by

$$L(f; E, d) = \sum_{i \geq 0} (-1)^i Trace(f^*|_{H^i(E,d)}),$$

where $f^*|_{H^i(E,d)}$ denotes the induced action of f on the i^{th} homology of the complex. It is a classical fact that $L(f; E, d)$ is intimately related to the fixed point submanifold of f. More precisely, the Lefschetz formula computes this class in terms of local data over the fixed point submanifold [2]. The link between the Lefschetz formulaes and index theory was first proved by Atiyah and Segal [5] who use the H-equivariant index theorem [4] as a tool to compute these Lefschetz classes. The starting point is that $L(f; E, d)$ can be viewed as the evaluation at f of the H-equivariant analytical index of the elliptic complex (E, d). The Atiyah-Singer index theorem together with a localization result then enables to deduce the Lefschetz fundamental theorem.

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This approach to fixed point formulaes works as well in the case of foliations. In [11,16] the index theorem for foliations was proved generalizing Atiyah-Singer's results. Let (V, g, F) be a smooth riemannian compact foliated manifold, the authors of [16] use Kasparov's bivariant theory to define an analytical index class for any longitudinal pseudodifferential operator elliptic along the leaves. This class lives in the K-theory group of the Connes' C^*-algebra associated with the foliation [12] and the main theorem of [16] is a topological computation of this index class giving as a byproduct the Atiyah-Singer index theorem for fibrations. Despite the case of fibrations where one has to compute the K-theory group of a compact space (the base), for general foliations, the problem of computing the K-theory of Connes' C^*-algebra is highly non trivial and is one of the topics of the Baum-Connes conjecture [15]. The differential geometry of the space of leaves is no more relevant in general and Connes' idea [15] is to use cyclic cohomology as an alternative to construct additive maps from the K-theory group, not necessarily of the C^*-algebra but of a dense smooth subalgebra, to the scalars. The Chern character construction becomes a particular case of this general pairing of cyclic cohomology with K-theory. A further step was then the cyclic index theorem for foliations which was proved in [14]. This theorem uses the cyclic cohomology of the regular algebra of the foliation to deduce a topological formula for the cyclic index *à la Atiyah-Singer*. In the case of holonomy-invariant transverse measures for instance, one recovers the measured index theorem [12].

If now f is an isometry of (V, g) which preserves the leaves of F, and H is the compact Lie group generated by f in the group of isometries $Iso(V, g)$, then the problem of existence of fixed points under the action of f has been solved in [6] in the whole generality following the method of [5]. Thus *Lefschetz classes* with respect to H-invariant *longitudinally* elliptic complexes [12] are defined in the localized H-equivariant K-theory of the Connes' C^*-algebra of the foliation. Just like in the classical setting, they bring the action of f on the homology of the H-invariant longitudinally elliptic complexes. The main result of [6] was then a Lefschetz theorem that expresses these classes in terms of topological data over the fixed-point submanifold of f.

Now again because the computation of the equivariant K-theory groups of foliation C^*-algebras is a hard problem in general, we naturally use in this paper cyclic cohomology as a more computable tool to construct cyclic Lefschetz numbers. Invariant currents [18] enable to explicit these numbers and to state a more useful fixed point formula. This is a generalization of the Connes' cyclic index theorem for foliations and some applications are treated in [7].

More precisely, we imitate the Chern-Connes construction and define a notion of equivariant cyclic cohomology which pairs with equivariant K-theory. This pairing behaves well with respect to localization at the prime ideal associated with f and enables to define the cyclic Lefschetz numbers. The non vanishing of all such numbers give criterions for the existence of fixed points under the action of f. Furthermore, when the fixed point submanifold V^f is non empty and transverse to the foliation, we obtain a basic Lefschetz formula that computes the Lefschetz numbers in terms of topological data over V^f (Theorem 9).

These results extend previous Lefschetz formulaes obtained by J. Heitsch and C. Lazarov in the case of holonomy invariant measures [20, 21].

We now describe the contents of each section. Section 2 gives a brief survey of the K-theory Lefschetz theorem. In section 3, we define the appropriate equivariant cyclic cohomology and construct for our isometry a pairing between this equivariant cyclic cohomology and equivariant K-theory which factors through the representation ring of the compact Lie group generated by the isometry. In section 4, we return to the foliated case and recall the more or less known between basic homology and cyclic cohomology which generalizes the well known isomorphism in the case of fibrations. In section 5, we show that basic homology furnishes equivariant cocycles with respect to longitudinal actions. In section 6, we use the results of Section 4 and Section 5 together with the K-theory Lefschetz theorem to prove the basic Lefschetz formula. In section 7 we translate the cyclic Lefschetz formula in cohomological terms using Connes' cyclic index theorem and we also state some other corollaries including a rigidity criterion.

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2. Review of the K-theory Lefschetz theorem.

Most of the results of this paper are based on the K-theory Lefschetz theorem and we recall the material from [6]. Let (V, F) be a C^∞-compact foliated manifold, this means that F is an integrable smooth subbundle of the tangent bundle $T(V)$. We will denote by G the holonomy groupoid of the foliation (V, F), and G^x will mean the subset of G which consists in the holonomy classes of paths with range $x \in V$. Let H be a compact Lie group of F-preserving diffeomorphisms of V. We fix once for all a smooth Haar system on G defined out of a Lebesgues-class measure along the leaves which by an averaging argument can be assumed H-invariant. In the same way, we can assume that the action of H preserves some metric on V and so acts by isometries. The C^*-algebra associated by A. Connes to the foliation, denoted $C^*(V, F)$ [11], is then an H-algebra in the sense that the induced action of H is defined by a strongly continuous morphism α from H to the automorphism group of $C^*(V, F)$.

Let P be an elliptic H-invariant order 0 compactly supported pseudodifferential G-operator acting on sections of the H-vector bundle E [12]. We choose an H-invariant hermitian structure on E and denote by $\epsilon_{V,E}$ the Hilbert C^*-module over $C^*(V, F)$, associated with the continous field of Hilbert spaces $L^2(G^x, s^*E, \nu^x)$. Then the couple $(\varepsilon_{V,E}, P)$ represents a class in the Kasparov equivariant group $KK^H(\mathbb{C}, C^*(V, F))$ called the analytical longitudinal H-index of P, and denoted by $Ind^H_{a,V}(P)$.

On the other hand, the H-submersion

$$p : F \to V/F$$

is K^H-oriented. Recall that a map from F to the space of leaves is by definition a G-valued 1-cocycle and that then the pull back bundle $p^*(\nu)$ of the transverse bundle ν to the foliation makes sense and is a vector bundle over F [?]. The K^H-orientation of p means that the vector bundle $T(F) \oplus p^*(\nu)$ admits a spin^c structure which is given by an H-equivariant hermitian vector bundle of irreducible representations of the Clifford algebra bundle associated with $T(F) \oplus p^*(\nu)$. See [17] for more precise definitions.

Using this spin^c-bundle A. Connes and G. Skandalis constructed in a topological way which generalizes the Atiyah-Singer shrieck maps, a Gysin class [17]

$$p! \in KK_H(C_0(F), C^*(V,F)).$$

This enables to define the topological longitudinal H-index of P by :

$$Ind^H_{t,V}(P) = [\sigma(P)] \otimes_F p!,$$

where $[\sigma(P)]$ is the H-equivariant class of the principal symbol of P in $K_H(F)$. The longitudinal H-equivariant index theorem can then be stated as follows:

Theorem 1. *[16] $Ind^H_{a,V}(P) = Ind^H_{t,V}(P)$.*

Note that we can easily get rid of the order 0 assumption on P by classical arguments. Note also that this theorem works as well for elliptic H-invariant G-complexes [4] just like in the "non foliated case".

Now we are in a position to state the K-theory Lefschetz theorem. Let f be a leaf-preserving isometry of V with respect to some fixed riemannian metric g on V. Let H be the closed subgroup generated by f in the compact Lie group of isometries of (V,g). H is then a compact Lie group and it preserves the longitudinal subbundle F. Let (E,d) be an elliptic G-complex over (V,F) which we assume to be H-invariant. Let $I_f = \{\chi \in R(H), \chi(f) = 0\}$ be the prime ideal associated with f in the representation ring $R(H)$. Localization of the ring $R(H)$ with respect to I_f gives a ring of fractions denoted $R(H)_f$. For an $R(H)$-module M, we shall denote by M_f its localization with respect to I_f and which is then an $R(H)_f$-module in an obvious way. The Lefschetz class of the foliated isometry f with respect to (E,d) is naturally defined by [6]:

Definition 1. The localized analytic H-index of (E,d) with respect to the above ideal I_f is called the Lefschetz class of f with respect to (E,d) and denoted by $L(f;E,d)$, so:

$$L(f;E,d) = \frac{Ind^H_{a,V}(E,d)}{1_{R(H)}} \in K^H(C^*(V,F))_f.$$

When the foliation is top dimensional say with one leaf the manifold V itself, (E,d) becomes a classical pseudodifferential elliptic H-invariant complex over the compact manifold V and the $R(H)_f$-module $K^H(C^*(V,F))_f$ coincides with $K^H(\mathcal{K}(L^2(V))_f$, where $\mathcal{K}(L^2(V))$ is the elementary C^*-algebra and $L^2(V)$ is defined using a Lebesgues measure that can be constructed out of local charts. Given any smooth kernel k, $\forall h \in H$ we have the smooth kernel $k^h : (x,y) \mapsto k(h^{-1}x, y)$. Now the integral of k^h against the diagonal of $V \times V$ gives exactly the Morita identification $K^H(\mathcal{K}(L^2(V)) \cong R(H)$. This shows the coherence of definition 1 with the classical one [4]. Extending the classical fixed point theorem we can state a longitudinal Lefschetz theorem which relates the Lefschetz class to the appropriate topological data over the fixed point submanifold V^f of f. We are only interested in the case where V^f is transverse to the foliation (this happens for instance when H is connected and preserves the leaves [20]). Let then $F^f = TV^f \cap F$ be the integrable subbundle of $(TV)^f = T(V^f)$, and let $i : F^f \hookrightarrow F$ be the K^H-oriented H-inclusion. The associated Gysin-Thom element $i! \in KK_H(F^f, F)$ is then well defined [6]. Recall that the H-equivariant K-theory of the H-trivial manifold F^f

is isomorphic to the tensor product $K(F^f) \otimes R(H)$ and that the index map for the foliated H-trivial manifold (V^f, F^f) is then given by

$$Ind^H_{V^f} \cong Ind_{V^f} \otimes R(H) : K_H(F^f) \cong K(F^f) \otimes R(H) \to$$
$$K^H(C^*(V^f, F^f)) \cong K(C^*(V^f, F^f)) \otimes R(H).$$

Theorem 2. *[6] With the above notations, the following triangle is commutative:*

$$\begin{array}{lcl} K_H(F^f) \cong K(F^f) \otimes R(H) & \stackrel{i!}{\to} & K_H(F) \\ Ind^H_{V^f} \cong Ind_{V^f} \otimes R(H) \searrow & & \downarrow Ind^H_V \\ & & K^H(C^*(V,F)) \end{array}$$

In the above theorem we have denoted by $Ind^H_{V^f}$ the H-equivariant longitudinal analytic (or topological thanks to the equivariant longitudinal index theorem recalled above) index $R(H)$-homomorphism for the compact foliated manifold (V^f, F^f) as recalled above. This index map rigorously takes values in the $R(H)$-module $K^H(C^*(V^f, F^f))$ and we have used the quasi trivial element of $KK^H(C^*(V^f, F^f), C^*(V, F))$ correspopnding to the Morita extension [16] to see its range in $K^H(C^*(V, F))$. Now we can state the K-theory Lefschetz theorem using the functorial properties of the localization at the prime ideal I_f:

Corollary 1. *(The K-theory Lefschetz theorem) Let (V, F) be a compact foliated manifold and let f be a diffeomorphism of V which preserves the longitudinal subbundle F. We assume that f belongs to a compact Lie group of diffeomorphisms of V and denote by H the compact Lie group generated by f. Then H also preserves the bundle F and with the above notations, the Lefschetz class of f with respect to an H-invariant elliptic G-complex (E, d) is given by:*

$$L(f; E, d) = (Ind_{V^f} \otimes R(H)_f) \left(\frac{i^*[\sigma(E, d)]}{\lambda_{-1}(N^f \otimes \mathbb{C})} \right)$$

where $i^* : K_H(F)_f \to K_H(F^f)_f$ is the restriction homomorphism, N^f is the normal H-vector bundle to V^f in V, and $\lambda_{-1}(N^f \otimes \mathbb{C}) = \sum(-1)^i[\Lambda^i(N^f \otimes \mathbb{C})]$ in $K_H(V^f)_f \cong K(V^f) \otimes R(H)_f$.

The above fraction means that $\lambda_{-1}(N^f \otimes \mathbb{C})$ is a unit in the ring $K_H(V^f)_f$ and we have used the $K_H(V^f)_f$-module structure of $K_H(F^f)_f$. Whence, $L(f; E, d)$ coïncides with the (naturally extended to $K(F^f) \otimes R(H)_f$) index of a virtual operator over the foliated fixed point manifold. This theorem is the exact generalization of the now classical Atiyah-Segal theorem [5]. As a simple consequence we get the following fixed point criterion:

Proposition 1. *Under the assumptions of Corollary 1, we have:*

$$(\exists (E, d) / L(f; E, d) \neq 0) \Rightarrow V^f \neq \emptyset.$$

When V^f is a strict transversal, ie: when V^f is transverse to the foliation with dimension exactly the codimension of the foliation, then we obtain:

$$L(f; E, d) = \frac{\sum(-1)^i[E^i|_{V^f}]}{\sum(-1)^j[\wedge^j(F|_{V^f} \otimes \mathbb{C})]} \otimes_{V^f} [[V^f]].$$

Where $[[V^f]]$ is the class associated to the transversal V^f in $KK^H(C(V^f), C^*(V,F))$ as defined in [11]. Kasparov product by $[[V^f]]$ corresponds to *K-integration over V^f*, so that the above formula agrees with the classical ones. In particular when (E,d) is the de Rham complex along the leaves, our Lefschetz class coincides with the *K-volume* of V^f, that is the class $[V^f]$, which is represented by a simple H-invariant idempotent in $C_c^\infty(G,\Omega^{1/2})$ [11].

To end this section, let us just point out that the H-equivariant index morphism can be defined at the $K^H(C_c^\infty(G,\Omega^{1/2}))$ level just like in the non equivariant case. This enables to use the cyclic cohomology (more precisely the H-equivariant one) of the locally convex algebra $C_c^\infty(G,\Omega^{1/2})$ to construct cyclic Lefschetz numbers. This is necessary to get scalar criteria on the existence of fixed points as well as explicit integrality results.

3. Equivarant cyclic cocycles

Our goal in this section is to define the notion of equivariant cyclic cocycles. The definition we have adopted here consists *grosso modo* in cyclic cocycles on the discrete product with a further twist assumption. This assumption is fulfilled in the applications we are interested in and insures that the pairing with equivariant K-theory will factor through the representation ring.

Moreover this definition reflects the notion of equivariant Fredholm modules and is then in the spirit of equivariant Kasparov's theory and equivariant Chern character.

Let $\mathcal{A}$ be a *-subalgebra of the C^*-algebra $L(\mathcal{H})$ where $\mathcal{H}$ is some fixed Hilbert space. Let Γ be a (discrete) group and let $U:\Gamma\to U(\mathcal{H})$ be a representation of Γ in the group of of unitaries of $\mathcal{H}$. Assume that the induced action of Γ on $L(\mathcal{H})$ preserves $\mathcal{A}$. We shall denote by $<\mathcal{A},\Gamma>$ the *-subalgebra of $L(\mathcal{H})$ generated by finite sums of terms of the form

$$a\circ U(\gamma), a\in\mathcal{A}\subset L(\mathcal{H}) \text{ and } \gamma\in\Gamma.$$

A k-Hochschild cochain on an algebra $\mathcal{B}$ is a $(k+1)$-linear form on $\mathcal{B}^{k+1}$. The space of such Hochschild cochains will be denoted by $C^k(\mathcal{B})$. A Hochschild k-cochain ϕ on $\mathcal{B}$ is cyclic if

$$\phi(b^k,b^0,...,b^{k-1}) = (-1)^k\phi(b^0,...,b^k).$$

Definition 2. (i) Let $\gamma\in\Gamma$ be a fixed element. A γ-equivariant k-cochain τ on $\mathcal{A}$ is a k-Hochschild cochain on $<\mathcal{A},\Gamma>$ such that $\forall(a^0,...,a^k)\in\mathcal{A}^{k+1}$ and $\forall j\in\{0,1,...,k\}$, we have:

$$\tau(a^0,..,a^j\circ U(\gamma),a^{j+1},...,a^k) = \tau(a^0,...,a^j,U(\gamma)\circ a^{j+1},...,a^k).$$

(ii) A Γ-equivariant k-cochain is a k-Hochschild cochain on $<\mathcal{A},\Gamma>$ which is γ-equivariant for every $\gamma\in\Gamma$.

Remark 1. In the whole paper we will only use the restrictions of our k-cochains to the union over j of the spaces

$$\mathcal{A}^j\otimes(<\mathcal{A},\Gamma>)\otimes\mathcal{A}^{k-j-1}.$$

Thus it it possible (but unnecessary for our applications here) to use a more precise definition.

Remark 2. In fact, when $\mathcal{A}$ is unital and τ is a Hochschild cocycle which is normalized in the sense that:

$$\tau(b^0, ..., b^k) = 0 \text{ whenever one of the } {b^j}'s \text{ is scalar,}$$

we have:

$$[\tau \text{ is } \gamma - \text{equivariant }] \Leftrightarrow \tau(b^0, ..., b^k) = 0 \text{ whenever one of the } {b^j}'s \text{ is } U(\gamma).$$

We will however use a non unital algebra for foliations.

Recall that the Hochschild coboundary $b : C^k(\mathcal{B}) \to C^{k+1}(\mathcal{B})$ is given by:

$$b\tau(b^0, b^1, ..., b^{k+1}) = b'\tau(b^0, b^1, ..., b^{k+1}) + (-1)^{k+1}\tau(b^{k+1}b^0, b^1, ..., b^k),$$

where

$$b'\tau(b^0, b^1, ..., b^{k+1}) = \sum_{0\leq j\leq k} (-1)^j \tau(b^0, ..., b^j b^{j+1}, ..., b^{k+1}).$$

If τ is a γ-equivariant k-cochain then $b(\tau)$ is a γ-equivariant $(k+1)$-cochain as can be proved easily. If τ is a cyclic k-cochain on $< \mathcal{A}, \Gamma >$ then τ is γ-equivariant if $\forall(a^0, ..., a^k) \in \mathcal{A}^{k+1}$ we have:

$$\tau(a^0 \circ U(\gamma), a^1, ..., a^k) = (-1)^k \tau(U(\gamma) \circ a^1, a^2, ..., a^k, a^0).$$

We shall denote by $C^*_{\lambda,\gamma}(< \mathcal{A}, \Gamma >)$ the graded complex vector space of γ-equivariant cochains which are cyclic on $< \mathcal{A}, \Gamma >$ (γ-cyclic cochains for short). Then we have:

Lemma 1. *(i) Denote by A the cyclic antisymetrization operator [15], then for any γ-cyclic cochain τ, $A\tau$ is a γ-cyclic cochain;*
(ii) $(C_{\lambda,\gamma}(< \mathcal{A}, \Gamma >), b)$ is a well defined subcomplex of the cyclic complex of $< \mathcal{A}, \Gamma >$.

The proof of this lemma is straightforward and is omitted here.

Thus, $H^{*,\gamma}_\lambda(< \mathcal{A}, \Gamma >)$ will denote the cohomology of the above subcomplex and we obviously have the map

$$J_\Gamma : H^{*,\gamma}_\lambda(< \mathcal{A}, \Gamma >) \to H^*_\lambda(< \mathcal{A}, \Gamma >),$$

which consists in forgetting the γ-equivariance. Many cyclic cocycles on crossed products do come from Γ-equivariant cyclic cocycles as can be seen by the following examples.

Examples. (1) The most important class of examples is furnished by the equivariant spectral triples that give rise to equivariant Fredholm modules: Let $(\mathcal{H}, D)$ be any (unbouded) (n, ∞)-summable Fredholm module over the Γ-*-algebra $\mathcal{A}$. We assume that Γ acts on $\mathcal{H}$ and that this action agrees with the original one on $\mathcal{A}$. We also assume that D is Γ-invariant (this can be improved to deal with Kasparov's theory). Then the analytic Chern character $\tau = ch(\mathcal{H}, D)$ is defined (in the even case) by [15]

$$\tau(a^0, ..., a^n) = Trace_s(\alpha a^0[F, a^1]...[F, a^n]).$$

Here α is the grading involution automorphism of $\mathcal{H}$ and for $T \in L^{1,\infty}(\mathcal{H})$,

$$Trace_s(T) = Trace(F(TF + FT))$$

is the regularized trace. F is the sign of D corrected if necessary to satisfy $F^2 = 1$. It is then clear that τ extends in a Γ-equivariant cyclic cocycle on $< \mathcal{A}, \Gamma >$.

When $\mathcal{A} = C^\infty(V)$ for a compact spin manifold V with $\mathcal{H} = L^2(V, \mathcal{S})$ the L^2-space of spinors and D the associated Dirac operator, we obtain an equivariant cyclic cocycle whenever the group Γ acts on V by spin-preserving diffeomorphisms in the sense of [3].

Another important case is that of the Diff-equivariant situation where a spectral triple using the "signature" operator has already been defined by Connes and Moscovici. Thus if the compact Lie group of isometries of the manifold commutes with the action of the group of diffeomorphisms considered, then we again obtain an equivariant cyclic cocycle. This example can be extended to actions of pseudogroups and foliations where the assumption on the group is that it preserves the longitudinal bundle F.

(2) For foliations, we can define many other equivariant cyclic cocycles in the following way: Let (V, F) be any compact foliated manifold and take $\mathcal{A} = C_c^\infty(G, \Omega^{1/2})$, $\mathcal{H} = \oplus_{x\in V} L^2(G^x, \nu^x)$. Let Γ be a group of leaf-preserving isometries of (V, F). Then it is almost always possible to construct (see section 5) an induced action Ψ of Γ on $\mathcal{H}$ that *preserves each* $L^2(G^x, \nu^x)$ and such that in addition

$$\forall \gamma \in \Gamma, \forall f \in C_c^\infty(G, \Omega^{1/2}), \Psi(\gamma) \circ f \in C_c^\infty(G, \Omega^{1/2}) \text{ and } \Psi(\gamma) \circ f \circ \Psi(\gamma^{-1}) = \gamma(f).$$

Lemma 4 shows that all the cyclic cocyles arising from basic currents are Γ-equivariant. The case of the trace associated with a holonomy invariant transverse measure Λ is particularly interesting.

From now on and in view of our interest in this paper, say Lefschetz formulaes for foliated isometries, we assume that Γ is a compact Lie group. We fix the Γ-algebra $\mathcal{A}$ with its embedding in $L(\mathcal{H})$ for a Γ-Hilbert space $\mathcal{H}$.

Lemma 2. *Assume that $\mathcal{A}$ is unital. Let u be any invertible Γ-invariant element of $\mathcal{A}$ and denote by $\rho(a) = uau^{-1}$ the interior automorphism induced by u. Then for any $\gamma \in \Gamma$ and any γ-equivariant cyclic cocycle τ of degree $n > 0$, there exists a γ-equivariant cyclic cochain φ on $< \mathcal{A}, \Gamma >$ such that*

$$\rho^*(\tau) - \tau = b\varphi.$$

Proof. We simply apply the proof in [16][page 325] and see that the cochain constructed in this way is γ-equivariant whenever τ is. More precisely, let $a \in \mathcal{A}$ be an H-invariant element, say

$$[a, U(h)] = 0, \forall h \in H.$$

Set for $((b^0, ..., b^{n-1}) \in < \mathcal{A}, \Gamma >$,

$$\tau_a(b^0, ..., b^{n-1}) = \tau(b^0, ..., b^{n-1}, a).$$

τ_a is then a Hochschild cochain which is γ-equivariant by direct inspectation. If $\delta_a(b) = ab - ba$ and

$$\delta_a^*(\tau)(b^0, ..., b^n) = \sum_{0\le i\le n} \tau(b^0, ..., \delta_a(b^i), ..., b^n),$$

then $\delta_a^*(\tau)$ is a γ-equivariant (cyclic) cochain, since $\delta_a(b^j \circ U(\gamma)) = \delta_a(b^j) \circ U(\gamma)$.

Following [16][page 325], we verify using $b\tau = 0$ that:

$$(-1)^n \delta_a^*(\tau) = A(b'\tau_a).$$

Thus we obtain $\delta_a^*(\tau) = b((-1)^n A\tau_a)$ with $A\tau_a$ a γ-equivariant cyclic cochain on $< \mathcal{A}, \Gamma >$.

Let now u be an invertible Γ-invariant element of $\mathcal{A}$. Then again following [16][page 325] and by classical equivariant K-theory arguments, we can assume that $u = e^a$ where a is Γ-invariant. In fact there exists $b \in M_2(\mathbb{C})$ such that $a = vbv^{-1}$ and hence e^a is well defined using the exponential in $M_2(\mathbb{C})$ and we obtain by straightforward computation:

$$\rho^*(\tau) - \tau = (b \circ A)(\sum_{k\geq 1} \frac{(\delta_a^*)^{k-1}(\tau_a)}{k!}),$$

and if $\rho_v^*(\varphi)(a^0, ..., a^n) = \varphi(va^0v^{-1}, ..., va^nv^{-1})$ then

$$\delta_a^*(\alpha) = (\rho_v^{-1})^* \circ \delta_b \circ \rho_v^*.$$

The proof is then complete since

$$(\rho_v^{-1})^* \circ \sum_{k\geq 1} \frac{(\delta_b^*)^{k-1}(\tau_a)}{k!} \circ \rho_v^*,$$

is γ-equivariant. □

Let X be any finite dimensional Γ-representation and let τ any γ-equivariant cyclic cochain (resp. cocycle) on $< \mathcal{A}, \Gamma >$, where $\gamma \in \Gamma$ is again a fixed element. Then $\tau \sharp Trace$ (see [13]) is a γ-equivariant cyclic cochain (resp. cocycle) on the algebra $[\mathcal{A} \otimes End(X)] \rtimes \Gamma$. If $V(\alpha)$ is the unitary of X corresponding to the element $\alpha \in \Gamma$ then:

$$(\tau \sharp Trace)([a^0 \otimes A^0] \circ [U(\alpha^0) \otimes V(\alpha^0)], ..., [a^n \otimes A^n] \circ [U(\alpha^n) \otimes V(\alpha^n)]) :=$$
$$\tau(a^0 \circ U(\alpha^0), ..., a^n \circ U(\alpha^n)) Trace(A^0 \circ V(\alpha^0) \circ ... \circ A^n \circ V(\alpha^n))$$

Theorem 3. *Assume that Γ is a compact abelian Lie group. Let τ be any even γ-equivariant cyclic cocycle over $< \mathcal{A}, \Gamma >$. Let X be any finite dimensional representation of Γ and let e be a Γ-invariant projection in $\mathcal{A} \otimes L(X)$. We set:*

$$\tau^\gamma(e) = (\tau \sharp Trace)(e \circ [U(\gamma) \otimes V(\gamma)], e, ..., e).$$

Then τ^γ induces an additive map from equivariant K-theory of $\mathcal{A}$ to the scalars. Moreover, this map only depends on the class of τ in $H_{\gamma,\lambda}(< \mathcal{A}, \Gamma >)$ and we obtain in this way a well defined pairing between $H_{\gamma,\lambda}(< \mathcal{A}, \Gamma >)$ and the γ-localized Γ-equivariant K-theory of $\mathcal{A}$:

$$\tau^\gamma : K^\Gamma(\mathcal{A})_\gamma \to \mathbb{C}.$$

Remark 3. Theorem 5 is of course available for any compact Lie group but taking into account the conjugacy classe of γ. However we will only use the abelian version in this paper.

Proof. Using classical K-theory techniques, we can assume that $\mathcal{A}$ is unital (with a trivial action of Γ on $\mathbb{C}$). Also we only need to show that if e, e' are Γ-conjugated, then $\tau^\gamma(e) = \tau^\gamma(e')$. By Lemma 2, there exists an odd γ-equivariant cyclic cochain φ such that

$$\tau^\gamma(e) - \tau^\gamma(e') = b\varphi(e \circ [U(\gamma) \otimes V(\gamma)], e, ..., e).$$

A direct computation then shows that all but one of the terms in $b\varphi$ disappear and we get:

$$b\varphi(e \circ [U(\gamma) \otimes V(\gamma)], e, ..., e) = \varphi(e \circ [U(\gamma) \otimes V(\gamma)], e, .., e).$$

The γ-equivariance of φ then yields,

$$\varphi(e \circ [U(\gamma) \otimes V(\gamma)], e, .., e) = -\varphi([U(\gamma) \otimes V(\gamma)] \circ e, e, .., e),$$

and since e is Γ-invariant, we obtain the result.

The above computation also shows that the pairing only depends on the γ-equivariant cyclic cohomology class of τ. Furthermore, the pairing respects the ideal $I_\gamma = \{\chi \in R(\Gamma), \chi(\gamma) = 0\}$, for:
If $V' : \Gamma \to U(X')$ is an other irreducible representation of Γ then we obviously have

$$\tau^\gamma(e \otimes X') = \tau^\gamma(e) Trace(V'(\gamma)).$$

$\square$

Remark 4. The above pairing $< \tau^\gamma, e >$ is in fact well defined provided the projection e is γ-invariant and the whole Γ-invariance is not needed. So this result is in fact a result on the subgroup generated by γ in Γ so that the assumption that Γ be abelian is not restrictive.

Remark 5. The pairing in the odd case can also be defined in the Toeplitz sense but is not needed here.

To finish this section, let us explain the link with the equivariant index problem. Let $(\mathcal{H}, F)$ be an even (bounded) p-summable Γ-invariant Fredholm module over $\mathcal{A}$, then to any Γ-invariant projection $e \in \mathcal{A} \otimes L(X)$ for some Γ-representation X, $e \circ [F \otimes id_X] \circ e$ acting on $e(H \times X)$ is a Γ-invariant Fredholm operator which anticommutes with the grading of $\mathcal{H}$. One can then consider the Γ-equivariant index of its positive part as an element of the representation ring $R(\Gamma)$ of the compact Lie group Γ. This defines

$$Ch_\Gamma(\mathcal{H}, F) : K^\Gamma(\mathcal{A}) \to R(\Gamma),$$

that generalizes the Chern-Connes map to the equivariant case.

Proposition 2. *Denote by $Ch(\mathcal{H}, F)$ the Γ-equivariant cyclic cocycle associated with $(\mathcal{H}, F)$ as in [15]. Denote by $ev_\gamma : R(\Gamma) \to \mathbb{C}$ the evaluation at (a conjugacy class of) a fixed arbitrary element $\gamma \in \Gamma$. Then we have:*

$$ev_\gamma \circ Ch_\Gamma(\mathcal{H}, F) = Ch(\mathcal{H}, F)^\gamma.$$

Proof. Let $e \in A \otimes L(X)$ be any Γ-projection with X a finite dimensional representation of Γ. Denote by P the Γ-invariant Fredholm operator $[e(F \otimes id_X)e]_+ : e_+(\mathcal{H}^+ \oplus X) \to e_-(\mathcal{H}^- \otimes X)$. Then $Q = [e(F \otimes id_X)e]_-$ is a parametrix for P modulo $L^p(\mathcal{H} \oplus X)$ which is in addition Γ-invariant.

By the equivariant Calderon formula (See lemma 3 below), we obtain:

$$< ev_\gamma \circ Ch_\Gamma(\mathcal{H}, F), e >= Ind^\Gamma(P)(f) =$$

$$Trace_{e_+(\mathcal{H}^+ \oplus X)}(U(\gamma) \circ (1 - QP)^p) - Trace_{\mathcal{H}^- \oplus X}(U(\gamma) \circ (1 - PQ)^p).$$

The computation of $[e - e(F \otimes id_X)e(F \otimes id_X)e]^p$ in the non equivariant case [16][page 277] enables then to conclude. $\square$

The following lemma is more or less known to experts and we give it for the benefit of the reader.

Lemma 3. *(Equivariant Calderon formula)*
Let $U : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation of the compact Lie group Γ. Let P, Q be two Γ-invariant operators on $\mathcal{H}$ such that $1 - QP$ and $1 - PQ$ belong the Schatten ideal $L^p(\mathcal{H})$ for some $p \geq 1$. Then

$$\forall \gamma \in \Gamma, Ind^\Gamma(P)(\gamma) = Trace(U(\gamma) \circ [1 - QP]^p) - Trace(U(\gamma) \circ [1 - PQ]^p).$$

Proof. The proof in the non equivariant case generalizes straightforward (See for instance [15][page 304]), so we will be sketchy. We set $S_0 = 1-QP$ and $S_1 = 1-PQ$. Let e and f be the projections on the eigenspace associated with 1 for S_0 and S_1 respectively. These are defined for instance using Cauchy integrals around 1. Then e and f are Γ-invariant projections and $\forall \gamma \in \Gamma$, $U(\gamma)$ preserves their domains. We also have

$$Qf = eQ, Pe = fP.$$

This enables to restrict the Fredholm operator P to $Im(1 - e)$ so that it realizes a Γ-equivariant isomorphism between $Im(1 - e)$ and $Im(1 - f)$. Hence:

$$Ind^\Gamma(P) = Ind^\Gamma(P|_{Ker(QP)}) = [e] - [f] \in R(\Gamma),$$

and thus,

$$\forall \gamma \in \Gamma, Ind^\Gamma(P)(\gamma) = Trace(U(\gamma)|_{Im(e)}) - Trace(U(\gamma)|_{Im(f)}).$$

On the other hand, in restriction to $Im(1 - e)$ and with values in $Im(1 - f)$, P intertwines S_0^n and S_1^n, for any $n \geq 1$ and so the conclusion. ☐

Corollary 2. *Let $(\mathcal{H}, F)$ be a Γ-equivariant Fredholm module as above, then we have:*

$$\forall \gamma \in \Gamma, Ch(\mathcal{H}, F)^\gamma(K^\Gamma(\mathcal{A})) \subset ev_\gamma(R(\Gamma)) = R(\Gamma)(\gamma).$$

This corollary is very precious in view of the integrality theorems. It shows for instance that when Γ is a finite cyclic group $\mathbb{Z}_n$,

$$Ch(\mathcal{H}, F)^\gamma(K^\Gamma(\mathcal{A})) \subset \mathbb{Z}[e^{2i\pi/n}].$$

For foliated involutions, we obtain the integrality of $Ch(\mathcal{H}, F)^\gamma$, compare [3,4].

4. Basic homology and transverse cyclic cocycles

In this section we recall the basic homology of foliations as well as its relation with the cyclic cohomology of $C_c^\infty(G, \Omega^{1/2})$ following results from [14]. Note that because all our bundles are twisted by a line bundle of longitudinal densities, our definition is a slight modification of the classical one (see [21] for cohomology and [18] for the dual notion of invariant currents). Basic homology (in fact cohomology) was first introduced by Reinhart, in a now classical paper at the end of the fifties, to give an approximation to the homology of the space of leaves. This homology has been very useful especially to the study of Riemannian foliations [1,17,21,22]. On the other hand, Connes introduced the regular convolution algebra of the foliation and showed in many situations that its cyclic cohomology is an alternative way of describing the de Rham homology of the space of the leaves [14,15].

We define below a correspondence from basic homology to cyclic cohomology which generalizes the well known isomorphism in the case of fibrations. This correspondence is related with the one described in [10] for characteristic classes of

foliations and also to the computations of [8]. We also recall the cyclic index theorem for foliations and its basic version. We point out that the results of this section are not new and are more or less implicit in Connes' work.

We denote by ν the transverse vector bundle of the foliation that is the quotient bundle $T(V)/F$. The restriction of this bundle to any leaf is flat thanks to the action of the holonomy pseudogroup. More precisely, any path γ which is drawn in a leaf of (V,F) induces a holonomy transformation from a little transversal at $s(\gamma)$ onto a little transversal at $r(\gamma)$ whose differential is well defined as a linear isomorphism that identifies $\nu_{s(\gamma)}$ with $\nu_{r(\gamma)}$. This defines an action of the holonomy groupoid G on ν, that is a functor from the little category G to the category of vector spaces here the fibres of ν. By transposing this action, we get an action of G on the dual bundle

$$\nu^* = \{\alpha \in T^*(V), \forall x \in V, \forall X \in F_x, < \alpha_x, X >= 0\};$$

and hence also an action on the exterior powers $\Lambda^k(\nu^*)$ for $0 \le k \le q$.

Definition 3. [14] A k-twisted basic current is a generalized section (say a $C^{-\infty}$-section) of the bundle $\Lambda^{q-k}(\nu^*) \otimes \mathbb{C}_\nu$ which is in the kernel of the longitudinal differential

$$d_F : C^{-\infty}(V, \Lambda^{q-k}(\nu^*) \otimes \mathbb{C}_\nu) \to C^{-\infty}(V, \Lambda^{q-k}(\nu^*) \otimes F^* \otimes \mathbb{C}_\nu).$$

We will denote by A_k the space of twisted basic currents.

In the above definition $\mathbb{C}_\nu$ denotes the orientation line bundle of the transverse bundle ν. Let us now fix for the rest of this section a transverse distribution $\mathcal{L}$ that is a supplementary subbundle to F in $T(V)$. Then any element of A_k defines a continuous (with respect to the natural Fréchet topology) linear form on basic forms twisted by the line bundle of longitudinal densities. This gives a Poincare map:

$$A_k \to (\Omega_B^k)^*$$

where Ω_B denotes the space of basic forms twisted by the longitudinal densities, it can be identified with the space of basic forms via a choice of a nowhere vanishing longitudinal density. Whence Ω_B^k is exactly the kernel of the longitudinal differential d_F acting on smooth sections of $\Lambda^k(\mathcal{L}^*) \otimes \Omega_F$ with Ω_F denoting the longitudinal densities.

Definition 4. [15] Let A be a $\mathbb{C}$-algebra. A k-precycle over A is a quadruple $(\Omega, d, \int, \theta)$ such that

(i) Ω is a graded algebra $\Omega = \Omega^0 \oplus ... \oplus \Omega^k$, so that the product sends $\Omega^j \otimes \Omega^{j'}$ into $\Omega^{j+j'}$, and such that $\Omega^0 = A$;

(ii) d is a graded differential of degree 1 on Ω

$$\forall (\omega, \omega') \in \Omega^j \times \Omega^{j'}, d(\omega\omega') = (d\omega)\omega' + (-1)^j \omega(d\omega');$$

(iii) $\theta \in \Omega^2$ (see remark 3 below) satisfies $d\theta = 0$ and

$$d^2(\omega) = [\theta, \omega], \forall \omega;$$

(iv) $\int : \Omega^k \to \mathbb{C}$ is a graded closed trace, ie:

$$\forall (\omega, \omega') \in \Omega^j \times \Omega^{k-j}, \int \omega\omega' = (-1)^{j(k-j)} \int \omega'\omega; \text{ and } \forall \alpha \in \Omega^{k-1}, \int d\alpha = 0.$$

A k-precycle over A will be called a k-cycle if in addition $\theta = 0$ so that $d^2 = 0$.

Remark 6. The condition $\theta \in \Omega^2$ is too strong and in many situations including foliations, it suffices to assume the existence of such a θ so that $d\theta$ and $[\theta, .]$ do make sense.

When $A = C_c^\infty(M)$ for a smooth manifold $M^{(m)}$, every twisted closed k-current on M say a generalized section of $\Lambda^{m-k}(T^*(M)) \otimes \mathbb{C}_M$ such that $dC = 0$ gives rise to a k-cycle over A. Other examples come from the study of finitely summable Fredholm modules over $\mathbb{C}$-algebras [15]. In the above case $A = C^\infty(M)$ when M is compact, such Fredholm modules are induced by pseudodifferential elliptic operators and the Atiyah-Singer theorem gives the twisted closed current that defines them.

Recall that A. Connes has given a curvature method that enables to associate to any k-precycle over A a k-cycle over a modification of A (See [15][page 229]).

Now given a k-cycle over A, we obtain a cyclic k-cocycle (the Chern-Connes character of the cycle) on A by setting:

$$\tau(a^0, ..., a^k) = \int a^0 da^1 ... da^k. \tag{1}$$

Let us now describe the precycle used to define the transverse fundamental class of a foliation [13]. If we denote by Ω^j the space $C_c^\infty(G, \Lambda^j(r^*(\nu^*)) \otimes \Omega^{1/2})$ then we obtain a graded algebra by setting

$$\forall (\omega, \omega') \in \Omega^j \times \Omega^{j'}, \forall x \in V, \forall \gamma \in G^x,$$

$$\omega\omega'(\gamma) = \int_{G^x} \omega(\gamma_1) \wedge \gamma_1(\omega'(\gamma_1^{-1}\gamma)).$$

Note that if $s(\gamma) = y$ and $s(\gamma_1) = y_1$ then $\omega'(\gamma_1^{-1}\gamma) \in \Lambda^{j'}(\nu_{y_1}^*) \otimes |\Lambda F_y|^{1/2} \otimes |\Lambda F_{y_1}|^{1/2}$ and $\gamma_1(\omega'(\gamma_1^{-1}\gamma))$ is then in $\Lambda^{j'}(\nu_x^*) \otimes |\Lambda F_y|^{1/2} \otimes |\Lambda F_{y_1}|^{1/2}$. Thus the wedge product is well defined and gives an element of $\Lambda^{j+j'}(\nu_x^*) \otimes |\Lambda F_{y_1}| \otimes |\Lambda F_y|^{1/2} \otimes |\Lambda F_x|^{1/2}$ whose integral is well defined.

On the other hand, we can define a transverse differential [13,15]

$$d_{\mathcal{L}} : C_c^\infty(G, \Lambda^j(r^*(\nu^*)) \otimes \Omega^{1/2}) \to C_c^\infty(G, \Lambda^{j+1}(r^*(\nu^*)) \otimes \Omega^{1/2}),$$

which satisfies all the properties required for a precycle [15, page 266] but which, unless the transverse bundle is flat does not satisfy $d_{\mathcal{L}}^2 = 0$. So we need to use the Connes curvature method. Now we have the following theorem which is implicit in Connes' work:

Theorem 4. *Let C be a closed twisted basic k-current on V. For $(f_0, ..., f_k) \in C_c^\infty(G, \Omega^{1/2})^{k+1}$, put $\omega = f_0 df_1 ... df_k \in M_2(C_c^\infty(G, r^*(\Lambda^k(\nu^*)) \otimes \Omega^{1/2}))$ where d is the modified differential obtained out of $d_{\mathcal{L}}$ by the Connes curvature method. Then we have:*

(i) $\omega^{22} = 0$;

(ii) The following formula defines a cyclic k-cocycle over the regular algebra $C_c^\infty(G, \Omega^{1/2})$,

$$\tau_C(f_0, ..., f_k) = < C, \omega^{11}|_V > .$$

Note that $\omega^{11} \in \Omega^k$ involves the curvature θ in a complicated way for $k \geq 2$. Its restriction to $V = G^{(0)}$ is then a smooth k-differential form twisted by the longitudinal densities.

Proof. (i) Straightforward computation. In particular one easily sees that ω^{11} does not depend on θ for $k \leq 1$.

(ii) Consider $(\Omega = \oplus_{0\leq j\leq k}\Omega^j, d_{\mathcal{L}})$ then the conditions (i) and (ii) in the definition of a precycle are satisfied and this is already proved in [15, page 229]. Recall that the gobal de Rham differential decomposes with respect to the non canonical (since F^* is not a subbundle of $T^*(V)$) isomorphism $T^*(V) \cong F^* \oplus \nu^*$ induced by $\mathcal{L}$ in the following way [1]

$$d = d_F + d_{\mathcal{L}} + \partial.$$

where d_F is the longitudinal differential and ∂ is a component of degree $(-1, 2)$ that brings the non integrability of the bundle $\mathcal{L}$. It is a well known fact that if θ_1 is the smooth section of $F \otimes \Lambda^2(\nu^*)$ defined by

$$\forall X, Y \in C^\infty(V, \mathcal{L}), \theta_1(\pi_\nu(X), \pi_\nu(Y)) = \pi_F([X, Y]),$$

where π_F is the projection onto F along $\mathcal{L}$ and π_ν is the identification $\mathcal{L} \cong \nu$, then ∂ is exactly the interior product by θ_1 or rather its F-component.

Note that although $d|_{\Omega^j} = d_F + d_{\mathcal{L}}$, $(d|_{\Omega^j})^2$ takes into account the component ∂. Computing $d_{\mathcal{L}}^2$ we obtain the following equality

$$d_{\mathcal{L}}^2 + d_F\partial + \partial d_F = 0.$$

We can therefore apply [15, page 267, Lemma 4] and the method that enables to construct a cycle out of a precycle so that we only have to prove that $\int : \Omega^k \to \mathbb{C}$ given by

$$\int \omega =< C, \omega|_V >,$$

satisfies condition (iv) in definition 4. Take $\omega \in \Omega^j$ and $\omega' \in \Omega^{k-j}$ then using a partition of unity on G we can assume that $Supp(\omega) \subset U_1 \times_\gamma U$ where $U_1 \cong W_1 \times T_1$ and $U \cong W \times T$ are distinguished open sets and $U \times_\gamma U \cong W_1 \times W \times T$ as in [12]. Then

$$\forall (v, t) \in U, (\omega\omega')(v, t) = \int_{W_1} \omega(w_1, w, t) \wedge (w_1, w, t)[\omega'(w, w_1, t_1)]dw_1,$$

where $t_1 \in T_1$ corresponds to $t \in T$ under the holonomy transformation induced by γ^{-1}. Now the action of C consists in integrating first in the leaf direction (this corresponds for fibrations to a graded trace [14]) and then applying C. Thus we get

$$< C, \omega\omega' >=< C|T, \int_{W_1\times W} \omega(w_1, w, t) \wedge (w_1, w, t)[\omega'(w, w_1, t_1)]dw_1 dw >$$

$$= (-1)^{j(k-j)} < C|T, \int_{W_1\times W} (w_1, w, t)[\omega'(w, w_1, t_1)] \wedge \omega(w_1, w, t)dw_1 dw >$$

$$= (-1)^{j(k-j)} < C|T, \int_{W_1\times W} (w_1, w, t)[\omega'(w, w_1, t_1)\wedge(w, w_1, t_1)(\omega(w_1, w, t))]dw_1 dw >$$

Now a variant of the Lebesgue theorem and because C is holonomy invariant $(d_F C = 0)$, we obtain

$$< C, \omega\omega' >= (-1)^{j(k-j)}$$

$$\int_{W_1\times W} < C|T, (w_1, w, t)[\omega'(w, w_1, t_1) \wedge (w, w_1, t_1)[\omega(w_1, w, t)]] > dw_1 dw$$

$$= (-1)^{j(k-j)} \int_{W_1\times W} < C|T_1, \omega'(w, w_1, t_1) \wedge (w, w_1, t_1)[\omega(w_1, w, t)] > dw_1 dw$$

$$= (-1)^{j(k-j)} < C, \omega' \omega > .$$

Note that the θ obtained here is not an element of Ω^2 but of $C_c^{-\infty}(G, r^*(\Lambda^2 \nu^*) \otimes \Omega^{1/2})$, the curvature method works as well [15].

Let now $\alpha \in \Omega^{k-1}$ and let us prove that $\int d_{\mathcal{L}} \alpha = 0$. Because C is holonomy invariant we see that $< C, [d_{\mathcal{L}} \alpha]|_V >=< C, d_{\mathcal{L}}(\alpha|_V) >$ with obvious notations. Then once again we localize to a distinguished chart in V and notice that integration over the plaques commutes with the transverse differential up to longitudinal differentials so that the two conditions $d_F(C) = 0$ and $dC = 0$ enable to conclude. $\square$

Using the local Connes' algebras associated with a good cover [14], one defines a presheaf and gets a tricomplex (C^*, b, B, δ) which computes the localized periodic cyclic cohomology of the foliation algebra. Here δ denotes the Cech boundary and (b, B) are the usual operators in the cyclic bicomplex [15]. In [14], Connes computed this localized cohomology and identified it with $H_\nu^*(V, \mathbb{C})$, the Cech cohomology of V twisted by the locally constant line lundle $\mathbb{C}_\nu$. Thus we have a well defined localization map

$$l : HP^*(C_c^\infty(G, \Omega^{1/2})) \to H_\nu^*(V, \mathbb{C}).$$

We finally mention that from the spectral sequence computation in [14], one easily deduces that for any closed basic current C, $l(\tau_C)$ is exactly the corresponding localization of currents viewed locally using Poincaré duality [14].

The important result that will be needed in section 7 is the Connes cyclic theorem and we proceed now to recall it. Let P be any elliptic pseudodifferential G-operator over the compact foliated manifold (V, F) [12]. Assume that F is orientable and oriented, then P is invertible modulo the algebra $C_c^\infty(G)$ and its analytic index is well defined in the K-theory group $K_0(C_c^\infty(G))$ as the image of P by a connecting map [12]. On the other hand, every cyclic cocycle on $C_c^\infty(G)$ gives rise to an additive map from K-theory to the scalars.

Theorem 5. *[14,15] Let $\varphi \in HP^{2k}(C_c^\infty(G))$ be any even cyclic cocycle. then the following Connes index formula holds*

$$Ind_\varphi(P) :=< \varphi, Ind(P) >=< l(\varphi) Ch[\sigma(P)] Td(F \otimes \mathbb{C}), [T(V)] >$$

This theorem is in fact a consequence of the K-theory index theorem proved in [16] together with the good behaviour of the localization map with respect to Morita equivalence (See [14]).

Corollary 3. *Let C be any closed basic current, then under the assumptions of theorem 7, we have*

$$Ind_C(P) :=< \tau_C, Ind(P) >=< Ch[\sigma(P)] Td(F \otimes \mathbb{C}), [C] >$$

In corollary 3 we have used the inclusion $\nu^* \subset T^*(V)$ and the longitudinal orientation to see $[C]$ in $H_*(V, \mathbb{C})$.

5. Basic Lefschetz numbers

Let again (V, F) be any compact foliated manifold. Let f be any leaf-preserving transformation of V which is isometric with respect to some metric g on V. Denote by H the compact Lie group generated by f in $Iso(V, g)$. One easily checks that H preserves the subbundle F so that each element of H brings any leaf to a leaf. We denote by $\Gamma(f) \subset \mathcal{R}(F)$ the graph of the isometry f, where $\mathcal{R}(F)$ is the groupoid

associated with the leaf equivalence relation. We will need the following assumption:

(H) There exists a smooth section $\varphi^f : \Gamma(f) \to G$ of the submersion (s, r) so that the holomony of $\varphi^f(x, fx)$ coincides with the action of f.

(H) is not too restrictive thanks to Pradines' density theorem. Note that when the holonomy groups are all trivial this section is simply the identity map and (H) is automatically satisfied but we prefered to take into account these holonomy groups. (H) is also canonically satisfied for general foliations when H is connected by using a continuous path from 1 to f in H. Finally if one denotes by $sat(V^f)$ the saturation of the fixed point submanifold V^f say the union of the leaves that intersect V^f, then φ^f is always well defined on $(sat(V^f), F)$. More precisely, if $x \in sat(V^f)$, $y \in V^f \cap L_x$ where L_x is the leaf through x, and if $\gamma \in G^x_y$, set $\varphi^f(x, fx) = (f\gamma^{-1}) \circ \gamma$, φ^f is well defined, smooth and only depends on x (This was pointed out to us by G. Skandalis). Using the K-theory Lefschetz theorem recalled in Section **??**, we know that the Lefschetz class lives in the saturation of the fixed points. However and since our goal is to prove an independently useful fixed point formula using cyclic cohomology, we will make the assumption (H).

We shall write for simplicity $\varphi^f(x)$ instead of $\varphi^f(x, fx)$. Let now E be an H-equivariant hermitian smooth vector bundle over V, then we define an endomorphism $\Psi^E(f) = \left(\Psi^E(f)_x\right)_{x\in V}$ of the C^*-module $\epsilon_{V,E}$ by:

$$\Psi^E(f)_x : L^2(G^x, s^*E, \nu^x) \to L^2(G^x, s^*E, \nu^x)$$
$$\Psi^E(f)_x(\xi)(\gamma) = f\xi(\varphi^f(f^{-1}x)(f^{-1}\gamma))$$

for $\xi \in L^2(G^x, s^*(E), \nu^x)$ and $\gamma \in G^x$.

Note that the same definition works for any element $h \in H$ that preserves the leaves and so for any element of the dense subgroup H_1 composed of powers of f. When $E = V \times \mathbb{C}$, $\Psi^E(f) = \Psi(f)$ is a multiplier of the C^*-algebra of the foliation which in addition preserves $C_c^\infty(G, \Omega^{1/2})$. More precisely, if $k \in C_c^\infty(G, \Omega^{1/2})$ denote by $k^{h,l}$ the kernel of $\Psi(h) \circ k$ and by $k^{h,r}$ the kernel of $k \circ \Psi(h)$, then we have

$$k^{h,r}(\gamma) = k((h\gamma) \circ \varphi^h(s(\gamma))) \text{ while } k^{h,l}(\gamma) = k(\varphi^h(h^{-1}r(\gamma)) \circ h^{-1}\gamma)).$$

Note also that Ψ^E is a well defined representation of the group H_1 in $\mathcal{H} = \oplus_{x\in V} L^2(G^x, s^*E, \nu^x)$ and the induced action on $C_c^\infty(G, \Omega^{1/2})$ agrees with the original action in the sense that

$$\Psi(h) \circ k \circ \Psi(h^{-1}) = h(k).$$

Whence the assumptions of Section 3 are fulfilled with $\mathcal{A} = C_c^\infty(G, \Omega^{1/2})$. The following lemma is clear if one keeps in mind the fact that any twisted basic current is holonomy invariant together with the assumption (H).

Lemma 4. *Let C be a twisted closed basic current and denote by τ_C the cyclic cocycle on $C_c^\infty(G, \Omega^{1/2})$ associated with C by theorem 6. Then τ_C is an H_1-equivariant cyclic cocycle for the action defined by Ψ.*

Proof. For holonomy invariant distributions lemma 4 is clear and the proof for measures easily generalizes to deal with distributions. Let us then restrict to the case where $\dim(C) > 0$ as a current. Let $\mathcal{L}$ be any H-equivariant transverse distribution constructed for instance using any H-invariant metric on V. Then the

differential $d_{\mathcal{L}}$ is the $(0,1)$ component of the de Rham differential in the decomposition $T(V) \cong F \oplus \nu$ induced by $\mathcal{L}$. Let $h \in H$ and let $k_0, k_1 \in C_c^\infty(G, \Omega^{1/2})$, denote as before respectively by $k^{h,l}$ and $k^{h,r}$, the smooth kernels associated with $\Psi(h) \circ k$ and $k \circ \Psi(h)$ for any $k \in C_c^\infty(G, \Omega^{1/2})$.

Assume for simplicity that F is orientable with a chosen orientation. Fix $\gamma \in G_y^x$ and following the notations of [12] let $\phi : U \times_\gamma V \to \mathbb{R}^p \times \mathbb{R}^p \times \mathbb{R}^q$ be a local trivializing chart of G around γ. If $k \in C_c^\infty(U \times_\gamma V, \Omega^{1/2})$ then its differential in the transverse direction can be described as follows: We use the trivialization φ to deduce a trivialization of $\mathcal{L}^*$ that identifies it with a copie of $T^*(\mathbb{R}^q)$ in $\mathbb{R}^n$. $k \circ \phi^{-1}$ is then a smooth function with compact support on the trivial groupoid $\mathbb{R}^n \cong \mathbb{R}^p \times \mathbb{R}^p \times \mathbb{R}^q$, where the last copie of $\mathbb{R}^q$ is the one corresponding to $\mathcal{L}$ in V. The differential $d_\dashv(k)$ of $k \circ \phi^{-1}$ in the transverse direction $\mathbb{R}^q$ yields an element of $C_c^\infty(\mathbb{R}^p \times \mathbb{R}^p \times \mathbb{R}^q, T^*(\mathbb{R}^q))$. Finally, $\phi^*(d_\dashv(k))$ is the differential $d_{\mathcal{L}}(k)$ in the direction $\mathcal{L}$. Since $d_{\mathcal{L}}(k)$ is defined globally as the projection of of the DeRham differential onto $\mathcal{L}^*$, the local expression is independent of the chart used to define it. Now, for a trivialization of G around $\varphi^h(h^{-1}r(\gamma))h^{-1}\gamma$, we chose as a local neighborhood $h^{-1}U \times_{\varphi^h(h^{-1}r(\gamma))h^{-1}\gamma} V$ which is diffeomorphic to $\mathbb{R}^p \times \mathbb{R}^p \times \mathbb{R}^q$ with again $\mathbb{R}^q$ a q-plane in $\mathbb{R}^n$ corresponding to $\mathcal{L}$ in V. If $\gamma' \in U \times_\gamma V$ with $\phi(\gamma') = (u', v', t')$ then $h^{-1}(\gamma') \in h^{-1}U \times_{h^{-1}\gamma} h^{-1}V$ and $\varphi^h(h^{-1}r(\gamma))$ induces a germ of holonomy that is exactly the action of h on $\mathcal{L}$ thanks to the assumption (H). This shows that the transverse copie of $\mathbb{R}^q$ in the trivialisation ϕ' of $h^{-1}U \times_{\varphi^h(h^{-1}r(\gamma))h^{-1}\gamma} V$ coincides with that of $U \times_\gamma V$ and hence $\pi_{\mathbb{R}^q}[\phi'(\varphi^h(h^{-1}r(\gamma'))h^{-1}\gamma')] = \pi_{\mathbb{R}^q}(\gamma')$. So

$$d_{\mathcal{L}}(k^{h,l})(\gamma) = d_{\mathcal{L}}(k)(\varphi^h(h^{-1}r(\gamma))h^{-1}\gamma), \forall \gamma \in G.$$

Now recall that

$$[k_0 * d_{\mathcal{L}}(k_1^{h,l})](\gamma) = \int_{\gamma_1 \in G^x} k_0(\gamma_1)\gamma_1[d_{\mathcal{L}}(k_1^{h,l})(\gamma_1^{-1}\gamma)].$$

Let $x \in V$ and assume that $Supp(k_0) \subset U \times_\gamma V$ where $U \cong W \times T_1$ and $x \in V \cong W' \times T$ where T_1 and T correspond to the decomposition induced by $\mathcal{L}$. The isometry h^{-1} transforms U onto $h^{-1}(U) \cong W''' \times T'$ with $T' = h^{-1}(T_1)$, in the same way we have $h^{-1}(V) = V" \cong W" \times T"$ with $T" = h^{-1}(T)$. Let us compute $[k_0 * d_{\mathcal{L}}(k_1^{h,l})](x)$ using the local trivializations, we obtain

$$[k_0 * d_{\mathcal{L}}(k_1^{h,l})](x \cong (w,t)) = \int_{w_0 \in W} k_0(w_0, w, t)(w_0, w, t)[d_{\mathcal{L}}(k_1^{h,l})(w, w_0, t_1)]dw_0$$

$$= \int_{w_0 \in W} k_0(w_0, w, t)(h^{-1}w, w, t)\{(w_0, h^{-1}w, t")(d_{\mathcal{L}}(k_1)(h^{-1}w, w_0, t_1))\}dw_0$$

$$= \int_{w_0 \in W} k_0^{h,r}(w_0, h^{-1}w, t")(h^{-1}w, w, t)[(w_0, h^{-1}w, t")(d_{\mathcal{L}}(k_1)(h^{-1}w, w_0, t_1))]dw_0$$

$$= h[(k_0^{h,r} * d_{\mathcal{L}}k_1)(h^{-1}w, t")] = [h^*(k_0^{h,r} * d_{\mathcal{L}}k_1)](x).$$

Thus we obtain the following equality

$$[k_0 * d_{\mathcal{L}}(k_1^{h,l})]|_V = [h^*(k_0^{h,r} * d_{\mathcal{L}}k_1)]|_V.$$

In the same way we show that

$$[k_0 * d_{\mathcal{L}}(k_1) * d_{\mathcal{L}}(k_2) * ... * d_{\mathcal{L}}(k_r^{h,l})](x) = [h^*(k_0 * d_{\mathcal{L}}k_1 * ... d_{\mathcal{L}}^{h,r}k_{r-1} * d_{\mathcal{L}}k_r)](x).$$

Now if C is any twisted basic current then it is invariant under holonomy and so

$< C, [h^*(k_0 * d_{\mathcal{L}} k_1 * ... d_{\mathcal{L}} k_{r-1}^{h,r} * d_{\mathcal{L}} k_r)]|_V > = < C, [k_0 * d_{\mathcal{L}} k_1 * ... d_{\mathcal{L}} k_{r-1}^{h,r} * d_{\mathcal{L}} k_r]|_V >$.

It remains to treat the terms involving the curvature θ in

$$k_0 * d_{\mathcal{L}}(k_1^{h,l}) * d_{\mathcal{L}}(k_2) * ... * d_{\mathcal{L}}(k_r).$$

These terms are of the form $k_0 * k_1^{h,l} * A$ where A contains factors with θ. Thus Lemma 4 and hence the above proof works again. □

Theorem 6. *Let X be any finite dimensional representation of H and let $\tilde{e} \in \widetilde{C_c^\infty(G, \Omega^{1/2})} \otimes L(X)$ be an H-invariant projection, $\tilde{e} = e \oplus \lambda$ with $\lambda \in L(X)$. Let τ be any basic cyclic cocycle on $C_c^\infty(G, \Omega^{1/2})$. Then the formula*

$$\tau^f(\tilde{e}) = (\tau \sharp Trace)(\Psi^{V\times X}(f) \circ e, e, ..., e)$$

induces an additive map on the equivariant K-theory of $C_c^\infty(G, \Omega^{1/2})$ which respects the localization at the prime ideal of f.

Proof. Using Theorem 5 it is sufficient to prove that all the basic cocycles are $\Psi(H_1)$-equivariant. But this is the content of Lemma 4. □

Assume for instance that (V, F) admits a holonomy invariant transverse measure Λ. Recall that this induces a trace on $C^*(V, F)$ [12] which is finite on $C_c^\infty(G, \Omega^{1/2})$. We will denote this 0-cocycle by τ_Λ. Because the Ruelle-Sullivan current associated with Λ is a basic current we deduce:

Corollary 4. *Let e be an H-invariant projection of $C_c^\infty(G, \Omega^{1/2}) \otimes L(X)$, where X is a finite dimensional unitary representation of H. Then the formula:*

$$\tau_\Lambda^f(e) = (\tau_\Lambda \sharp Tr)\left(\Psi^{V\times X}(f) o e\right)$$

induces an additive map $\tau_\wedge^f : K^H(C^*(V, F))_f \to \mathbb{C}$.

Corollary 4 was first proved in [6] but is here a corollary of theorem 8 since τ_Λ is an H-equivariant cyclic 0-cocycle. If we define now the Lefschetz Λ-number of f to be:

$$L_\Lambda(f; E, d) = \tau_\Lambda^f(L(f; E, d)),$$

then we get out of the K-theory Lefschetz theorem a measured Lefschetz theorem which recovers the results of [19] when the diffeomorphism is isometric. In this measured case it is easy to see that $L_\Lambda(f; E, d)$ coincides with the alternate sum of the actions of f on the kernels of the laplacians of the G-complex so that when $F = T(V)$ we obtain the classical Lefschetz theorem.

Definition 5. Let C be any closed basic current and let τ_C be the H-equivariant cyclic cocycle on $C_c^\infty(G, \Omega^{1/2})$ associated with C by theorem 6. Then the C-Lefschetz number of f with respect to a pseudodifferential H-invariant G-complex (E, d) is defined by

$$L_C(f; E, d) := \tau_C^f(L(f; E, d)).$$

Let us point out that since V^f is a transverse compact submanifold, its saturation $\text{sat}(V^f) = U$ is an open submanifold of V and we can define very easily additive maps from the localized equivariant K-theory of $C_c^\infty(G(U), \Omega^{1/2})$ to $\mathbb{C}$ without any

H-equivariance assumption. In fact, we have an algebraic H-equivariant Morita equivalence

$$C_c^\infty(G(U), \Omega^{1/2}) \sim C_c^\infty(G_{V^f}^{V^f}, \Omega^{1/2})$$

and an isomorphism

$$K^H(C_c^\infty(G_{V^f}^{V^f}, \Omega^{1/2}) \cong K(C_c^\infty(G_{V^f}^{V^f}, \Omega^{1/2})) \otimes R(H).$$

We can then define $\tau^f = [\tau_* \otimes ev_f] \circ r_*$, where $\tau_* : K(C_c^\infty(G_{V^f}^{V^f}, \Omega^{1/2})) \to \mathbb{C}$ is defined by the same construction. This shows that in the almost H-trivial case (saturation of V^f) the maps we have constructed agree with the expected ones and one needs no assumption neither on τ nor on the existence of φ^f to construct such maps. Note also that our definition of τ^f coincides with $\tau_* \otimes ev_f$ whenever the action of H is trivial.

Theorem 7. *Let (E,d) be an H-invariant elliptic G-complex over (V,F). Let C be a closed twisted basic k-current on V and let τ_C be the cyclic k-cocycle over $C_c^\infty(G, \Omega^{1/2})$ associated with C in theorem 6. Then:*

$$L_C(f; E, d) \neq 0 (\Rightarrow L(f; E, d) \neq 0) \Rightarrow Ind_V^H(E, d) \neq 0 \textit{ and } V^f \neq \emptyset.$$

Proof. This is an immediate consequence of the K-theory Lefschetz theorem together with the additivity of $(\tau_C)^f$. □

6. The basic Lefschetz theorem

Let C be any closed twisted basic k-current on (V, F) and let $\tau = \tau_C$ be the associated H-equivariant cyclic k-cocycle on $C_c^\infty(G, \Omega^{1/2})$. We shall only consider the even case as explained before. Such currents are transverse in the sense that:

Lemma 5. *For every transverse submanifold W, let τ_C^W be the cyclic k-cocycle over $C_c^\infty(G(W), \Omega^{1/2})$ defined in theorem 6 but fo the foliated manifold W. Then, the following diagram commutes:*

$$\begin{array}{ccc} K_0(C_c^\infty(G(W), \Omega^{1/2})) & \xrightarrow{\varphi} & K_0(C_c^\infty(G, \Omega^{1/2})) \\ (\tau_C^W)_* \searrow & & \swarrow (\tau_C)_* \\ & \mathbb{C} & \end{array}$$

where G(W) is the holonomy groupoid of the restricted foliation and φ is Morita extension [16].

Proof. Let N be an open tubular neighborhood of W in V so that the fibres are included in the leaves of (V, F) and let us identify as usually N with the normal vector bundle to W in V. Recall that the Morita extension is the composition of a Mischenko map $C_c^\infty(G(W), \Omega^{1/2}) \to C_c^\infty(G(N), \Omega^{1/2})$ with the extension $C_c^\infty(G(N), \Omega^{1/2}) \to C_c^\infty(G, \Omega^{1/2})$. Thus thanks to the excision property, we only need to prove that the basic cocycles do commute with any Mischenko map. In [14] it is proved that for vector fibrations the curvature θ of the vector bundle N is killed in the computations. Now if $f \in C_c^\infty(N)$ satisfies

$$\forall w \in W, \int_{N_w} |f(n)|^2 d\beta(n) = 1,$$

(for a fiberwise measure β) then it realizes a Mischenko map by setting [11]:

$$M(\varphi)(\tilde{\gamma}) = f(s(\tilde{\gamma}))f(r(\tilde{\gamma}))\varphi(\gamma).$$

In this formula, $\tilde{\gamma} \in G(N)$ and γ is its projection in $G(W)$.

Let C be a closed twisted basic current and let τ_C be the associated k-cyclic cocycle. Let $\varphi \in C_c^\infty(G(W), \Omega^{1/2})$, then the transverse differential satisfies

$$d_{\mathcal{L}}(M(\varphi))(\tilde{\gamma}) = f(s(\tilde{\gamma}))f(r(\tilde{\gamma}))d_{\mathcal{L}}(\varphi)(\gamma) + f(s(\tilde{\gamma})\varphi(\gamma)(d_{\mathcal{L}}f)(r(\tilde{\gamma})).$$

Now $\forall w \in W$,

$$2\int_{N_w} f(n)d_{\mathcal{L}}f(n) = d_{\mathcal{L}}\int_{N_w} f(n)^2 = 0. \tag{2}$$

Thus if $(\varphi_0, ..., \varphi_k) \in C_c^\infty(G(W), \Omega^{1/2})^{k+1}$, then we obtain

$$M(\varphi_0) * d_{\mathcal{L}}(M(\varphi_1)) * ... * d_{\mathcal{L}}(M(\varphi_k)))(\tilde{\gamma}) =$$

$$\int_{\tilde{\alpha_0}\tilde{\alpha_1}...\tilde{\alpha_k}=\tilde{\gamma}} \varphi_0(\alpha_0)\tilde{\alpha_0}(d_{\mathcal{L}}(M(\varphi_1))(\tilde{\alpha_1}))\tilde{\alpha_0}\tilde{\alpha_1}(d_{\mathcal{L}}(M(\varphi_2)(\tilde{\alpha_2}))...)\tilde{\alpha_0}...\tilde{\alpha_{k-1}}(d_{\mathcal{L}}(M(\varphi_k)(\tilde{\alpha_k})).$$

All the terms arising from the transverse differential of f are annihilated by the condition (2) above and we obtain the result. $\square$

Now we can state using the notations recalled in Section 2:

Theorem 8. *(Basic Lefschetz theorem)*

Under the above assumptions, let (E, d) be an elliptic H-invariant G-complex over (V, F). Assume that the fixed-point submanifold is transverse to the foliation. Then, for every even closed twisted basic current C on (V, F), we have with $\tau = \tau_C$:

$$L_\tau(f; E, d) = Ind_{\tau, V^f}\left(\frac{i^*[\sigma(E, d)](f)}{\lambda_{-1}(N^f \otimes \mathbb{C})(f)}\right)$$

where $Ind_{\tau,V^f} = \tau_* \circ Ind_{V^f} : K(F^f) \otimes \mathbb{C} \to \mathbb{C}$ is the naturally extended Connes' cyclic index associated with τ, and $i : F^f \hookrightarrow F$ is the H-inclusion.

Proof. We will use theorem 3, which can be summarized by the commutativity of the following square

$$\begin{array}{ccc} K_H(F^f)_f & \stackrel{(i!)_f}{\rightarrow} & K_H(F)_f \\ Ind_{V^f \otimes R(H)_f} \downarrow & & \downarrow (Ind_V^H)_f \\ K^H(C_c^\infty(G(V^f, F^f), \Omega^{1/2}))_f & \stackrel{(j!)_f}{\rightarrow} & K^H(C_c^\infty(G, \Omega^{1/2}))_f \end{array}$$

and the fact that $(i!)_f$ is an $R(H)_f$-isomorphism with inverse $\frac{i_f^*}{\lambda_{-1}(N^f \otimes \mathbb{C})}$.

Recall that $j! : K^H(C_c^\infty(G(V^f, F^f), \Omega^{1/2})) \to K^H(C_c^\infty(G, \Omega^{1/2}))$ is the Morita extension as described in [16]. We deduce:

$$L(f; E, d) = (j!)_f \circ (Ind_{V^f} \otimes R(H)_f)\left(\frac{i^*[\sigma(E, d)]}{\lambda_{-1}(N^f \otimes \mathbb{C})}\right)$$

On the other hand, we have defined $L_\tau(f; E, d)$ as the image of $L(f; E, d)$ by the additive map

$$(\tau^f)_f : K^H(C_c^\infty(G, \Omega^{1/2}))_f \to \mathbb{C}.$$

So, we get:

$$L_\tau(f;E,d) = (\tau^f \circ j!)_f \circ (Ind_{V^f} \otimes R(H)_f) \left(\frac{i^*[\sigma(E,d)]}{\lambda_{-1}(N^f \otimes \mathbb{C})} \right).$$

Let us prove now that τ^f commutes with the Morita extension $j!$, say that:

$$\tau_V^f \circ j! = \tau_{V^f}^f.$$

This is a consequence of the H-equivariance of the Mischenko projection. More precisely:

Let N be an H-stable tubular neighborhood of V^f in V which is diffeomorphic to N^f. We denote as before on F^f and F^N the foliations inherited by V^f and N from the original foliation (V,F). Recall that the Haar system ν on G is given by the pullback $s^*(\alpha)$ of a Lebesgue measure α on the leaf manifold $\mathcal{F}$ (say: V with a discrete transverse topology). Let the restriction of α to N be the product of a leaf-measure α^f on $\mathcal{F}(V^f)$ by a fiberwise Lebesgue measure β, and choose $f_0 \in C_c^\infty(N)$, ≥ 0 such that:

$$\forall w \in V^f, \int_{N_w} |f_0(n)|^2.d\beta(n) = 1.$$

Then with $f \in C_c^\infty(N)$ given by:

$$f(n) = \frac{\int_H f_0(h^{-1}.n).dh}{\left[\int_{N_{\pi(n)}} |\int_H f_0(h^{-1}.n).dh|^2.d\beta(n)\right]^{1/2}}.$$

we have

$$\int_{N_w} |f(n)|^2.d\beta(n) = 1, \forall w \in V^f, f(h^{-1}.n) = f(n), \forall h \in H, \forall n \in N.$$

The Morita isomorphism $K_H(C_c^\infty(G(V^f,F^f),\Omega^{1/2})) \cong K_H(C_c^\infty(G(N,F^N),\Omega^{1/2}))$ is then induced by the tensor product with any H-invariant rank 1 projection, and is easily described by:

$$\varphi^H : C_c^\infty(G(W)) \to C_c^\infty(G(N))$$
$$\varphi^H(\xi)(\tilde\gamma) = \xi(\gamma).f(s(\tilde\gamma)).\overline{f(r(\tilde\gamma))}.$$

where γ is the projection of $\tilde\gamma$ under $\pi : G(N,F^N) \to G(V^f,F^f)$. $\varphi^H(\xi)$ is well defined and <u>compactly supported</u>, and it brings C_c^∞-half densities into C_c^∞-half densities so that using lemma 5 we only need to prove that:

$$\forall h \in H, \Psi(h) \circ \varphi^H(\xi) = \varphi^H(\Psi(h) \circ \xi).$$

But a straighforward computation of the kernels yields to the same result.

Now, we obtain

$$L_\tau(f;E,d) = (\tau_{V^f}^f)_f \circ (Ind_{V^f} \otimes R(H)_f) \left(\frac{i^*[\sigma(E,d)]}{\lambda_{-1}(N^f \otimes \mathbb{C})} \right)$$

and so,

$$L_\tau(f;E,d) = ([\tau_{V^f} \circ Ind_{V^f}] \otimes ev_f) \left(\frac{i^*[\sigma(E,d)]}{\lambda_{-1}(N^f \otimes \mathbb{C})} \right)$$

where $\mu_f : R(H)_f \to \mathbb{C}$ is the evaluation at f, given by $\mu_f(\chi/\rho) = \chi(f)/\rho(f)$.
We can also define the evaluation

$$K_H(F^f)_f \cong K(F^f) \otimes R(H)_f \to K(F^f) \otimes \mathbb{C},$$

that we denote by $x \mapsto x(f)$. Hence if Ind_{τ, V^f} is the extension of $\tau_{V^f} \circ Ind_{V^f}$ to the tensor product by $\mathbb{C}$, then we get:

$$L_\tau(f; E, d) = Ind_{\tau, V^f}\left(\frac{i^*[\sigma(E,d)](f)}{\lambda_{-1}(N^f \otimes \mathbb{C})(f)}\right).$$

$\square$

7. Some applications

There are several interesting corollaries of theorem 10 [2,19,20] but this work is too long to be inserted in the present paper. Let us just develop this basic Lefschetz theorem under the Chern character. So assume that F^f is oriented and let l^f be the localization map in the twisted homology by the transverse orientation of V^f, defined for the compact foliated manifold (V^f, F^f). Then $l^f(\tau_C|_{V^f}) = C|_{V^f}$ viewed in $H_{*,\nu}(V^f, \mathbb{C})$ as we have already mentioned and Connes' cyclic index theorem yields:

Theorem 9. *(Cohomological basic Lefschetz formula)*
Under the assumptions of Theorem 10 and for any closed twisted basic current C we have:

$$L_C(f; E, d) = \left\langle \frac{ch_{\mathbb{C}}(i^*[\sigma(E,d)](f))}{ch_{\mathbb{C}}(\lambda_{-1}(N^f \otimes \mathbb{C})(f))}.Td(F^f \otimes \mathbb{C}), C|_{V^f}\right\rangle$$

where Td denotes the Todd characteristic class.

Proof. Let ϕ_{C,V^f} be the map from $H^*(F^f, \mathbb{R})$ to $\mathbb{C}$ given by:

$$\phi_{C,V^f}(x) = \left\langle x.Td(F^f \otimes \mathbb{C}), C|_{V^f}\right\rangle.$$

Then, the Connes' index theorem for the cyclic cocycle associated with C is (up to a sign that we include in the definition of the Chern character) exactly the equality:

$$Ind_{C,V^f} = \phi_{C,V^f} \circ Ch.$$

The chern character can be extended to $K(F^f) \otimes R(H)_f$, with values in $H^*(F^f, \mathbb{C})$ by the map θ given by:

$$\theta(x \otimes \frac{\chi}{\rho}) = ch_{\mathbb{C}}\,[(x \otimes \chi/\rho)(f)] = ch_{\mathbb{C}}\,[(x \otimes \chi(f)/\rho(f))] = [\chi(f)/\rho(f)].ch(x).$$

Hence, if we trivially extend ϕ_{C,V^f} to $H^*(F^f, \mathbb{C})$, then:

$$\begin{aligned}(\phi_{C,V^f} \circ \theta)(x \otimes \chi/\rho) &= \left\langle ch(x).\tfrac{\chi(f)}{\rho(f)}.Td(F^f \otimes \mathbb{C}), C|_{V^f}\right\rangle \\ &= \left\langle ch_{\mathbb{C}}((x \otimes \tfrac{\chi}{\rho})(f)).Td(F^f \otimes \mathbb{C}), C|_{V^f}\right\rangle\end{aligned}$$

On the other hand,

$$L_C(f; E, d) = Ind_{C,V^f}\left(\frac{i^*[\sigma(E,d)](f)}{\lambda_{-1}(N^f \otimes \mathbb{C})(f)}\right)$$

and

$$(\phi_{C,V^f} \circ \theta)(u) = Ind_{C,V^f}(u(f)).$$

So we obtain:

$$L_C(f; E, d) = (\phi_{C,V^f} \circ \theta)\left(\frac{i^*[\sigma(E,d)](f)}{\lambda_{-1}(N^f \otimes \mathbb{C})(f)}\right)$$

and finally,

$$L_C(f;E,d) = \left\langle \frac{ch_{\mathbb{C}}(i^*[\sigma(E,d)](f))}{ch_{\mathbb{C}}(\lambda_{-1}(N^f \otimes \mathbb{C})(f))}.Td(F^f \otimes \mathbb{C}), C|_{V^f} \right\rangle$$

□

We point out that the term $ch_{\mathbb{C}}(i^*[\sigma(E,d)](f))$ is not easy to compute in the whole generality and the simplifications that occur in the geometric cases are very precious [4]. On the other hand, our theorem simplifies when the fixed point submanifold V^f is a strict transversal in the sense that it is transverse to the foliation with dimension exactly the codimension of the foliation. This corresponds in the non foliated case to isolated fixed points and we get:

Corollary 5. *If V^f is a strict transversal, then under the asumptions of Theorem 10 and for any closed twisted basic current C, we have the following Lefschetz formula:*

$$L_C(f;E,d) = \left\langle \frac{\sum_i (-1)^i ch_{\mathbb{C}}([E^i|_{V^f}](f))}{\sum_j (-1)^j ch_{\mathbb{C}}([\Lambda^j(F|_{V^f} \otimes \mathbb{C})](f))}, [C|_{V^f}] \right\rangle.$$

Proof. Here $F^f \cong V^f$, so:

$$i^*[\sigma(E,d)] = \sum_i (-1)^i [E^i|_{V^f}],\ N^f \cong F|_{V^f} \text{ and } Td(F^f \otimes \mathbb{C}) = Td(V^f \times \mathbb{C}) = 1$$

and hence, the corollary is proved. □

If (E,d) is the G-complex associated with the De Rham complex along the leaves, then only the zero component of the twisted basic current is involved and we have

$$L_C(f; deRham) = \langle 1, [C_0] \rangle$$

where 1 is the constant function on V^f, and $[C_0]$ is the 0-component of $[C|_{V^f}]$.

This shows that if $[C_0]$ is a positive measure, then $L_C(f; DeRham) \geq 0$. This is true for instance when we consider a holonomy invariant positive transverse measure, compare with [20]. We also deduce when the foliation is transversally oriented and with $C = [V/F]$ the transverse fundamental class,

$$L_{[V/F]}(f; deRham) = 0.$$

Now, all the computations of [4] can be rewritten here by replacing the characteristic classes by their power series. Let us just give the formula obtained:

Proposition 3. *We keep the assumptions of 3.3. The normal bundle N^f decomposes under the orthogonal action of H into:*

$$N^f = N^f(-1) \oplus \sum_{0<\theta<\pi} N^f(\theta)$$

Let $s(\theta) = dim(N^f(\theta)) \in H^0(V^f, \mathbb{Z})$, let $x_1, \ldots, x_{[s(-1)/2]}$ be the standard characters which generate the Pontryagin dual of the maximal torus of $O(s(-1))$. Let $y_1, \ldots, y_{s(\theta)}$ be the corresponding characters for $U(s(\theta))$. We set:

$$\begin{cases} \mathcal{R} = \sum \mathcal{R}_r(p_1, \ldots, p_r) = \left[\prod_{j=1}^{[s(-1)/2]} ((1+e^{x_j})/2).((1+e^{-x_j})/2)\right]^{-1} \\ \mathcal{S} = \sum \mathcal{S}_r^\theta(c_1, \ldots, c_r) = \left[\prod_{j=1}^{s(\theta)} \frac{1-e^{y_j+i\theta}}{1-e^{i\theta}}.\frac{1-e^{-y_j-i\theta}}{1-e^{-i\theta}}\right]^{-1} \end{cases}$$

where p_i is the ith symetric function of the x_i's (ie: a Pontryagin class) and c_i is the ith symetric function of the y_i's (ie: a Chern class). We get:

$$L_C(f;E,d) = \left\langle \frac{ch_{\mathbb{C}}(i^*[\sigma(E,d)](f))}{det(1-f|_{N^f})}. \prod_{0<\theta<\pi} \mathcal{S}^\theta(N^f(\theta)).\mathcal{R}(N^f(-1)).Td(F^f \otimes \mathbb{C}), [C|_{V^f}] \right\rangle$$

Because of its particular interest in relation with the Riemann-Roch theorem for foliations, we give the formula obtained for the Dolbeault complex:

Corollary 6. *We suppose here that the leaves are holomorphic manifolds. If $(E,d) = Dolbeault(X)$ is the G-complex associated with the Dolbeault complex along the leaves with coefficients in an analytic vector bundle X then:*

$$L_C(f; Dolbeault(X)) = \left\langle \frac{ch([X|_{V^f}](f)}{det(1-f|_{N^{f,*}})}.Td(F^f), C|_{V^f}) \right\rangle$$

Proof. See [4]. □

Remark 7. The good definition of the transverse fundamental spectral triple yields to very precious integrality results by the use of similar ideas. It is an interesting challenge to work out the details of this kind of integrality results.

In the same way we obtain the expected usual characteristic classes for the signature operator along the leaves of an oriented foliation as well as for the Dirac operator when the foliation is K-oriented. The signature case is more relevant in relation with the integrality theorems which are too important to be quickly treated here. For the Dirac operator along the leaves, we can already generalize the Atiyah-Hirzebruch's rigidity theorem [3] in the lines of [20].

Assume that F is an even dimensional spin vector bundle. Then, we have a well defined Dirac operator D along the leaves, and we can apply the Lefschetz theorem to D whenever it is preserved by H. Let $\hat{A}(F)$ be the $\hat{A}$-characteristic class of the vector bundle F [3], and let τ_C be any basic k-cocycle over $C_c^\infty(G, \Omega^{1/2})$, the cyclic longitudinal $\hat{A}$-genus is then defined by

$$\hat{A}_\tau(F) = \left\langle \hat{A}(F), [C] \right\rangle.$$

Using the cyclic Lefschetz theorem, we can generalize Proposition 3.2 of [20] by, first taking into account the eventual relevant holonomy, second dealing with all the basic cocycles which are traces in the sense of [13]. Here we only state the result in the case of the transverse fundamental class $\tau_C = [V/F]$ [13]. See [7] for the general case and also for the details.

Proposition 4. *If (V,F) is transversally oriented, then*

$$\hat{A}_{[V/F]}(F) \neq 0 \Rightarrow \left\{ \begin{array}{c} \textit{no compact connected Lie group can act non} \\ \textit{trivially on } V \textit{ as a group of isometries of} \\ V \textit{ preserving the leaves and the spin structure} \end{array} \right.$$

The proof of this proposition uses the continuity of the map $f \to L_{[V/F]}(f;E,d)$ and a generalisation of the method of [3]. Note that when the group is connected and preserves each leaf, the fixed point submanifold is necessarily transverse to F [20].

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